Math 540 Extra Homework 1 Name:________________________

Question 1 Let G be a finitely generated group. Let $S = \{g_1, g_2, \ldots, g_k\}$ be a finite system of generators. Thus any element $g \in G$ can be written a product $g = g_{k_1}^{r_1} g_{k_2}^{r_2} \cdots g_{k_\ell}^{r_\ell}$, where $r_i \in \mathbb{Z}$ of elements in the generating set S. Define a pre-norm on G, by $|||g||| = \sum |r_i|$. Since an element $g \in G$ can have multiple representations as a product of elements of the generating set, we define a norm on G by

$$\|g\| = \min_{\text{representaions of g}} |||g|||$$

Note that if $e \in G$ is the identity element, then $\|e\| = 0$. Define a metric on G by setting $d(g,h) = \|gh^{-1}\|$.

a) Show that this defines a metric on G (that depends on generating set S.) This metric is called the *word metric* on G, and it turns out that study of G as a metric space can lead to algebraic information about the group G.

b) Let $\phi(r)$ denote the number of elements in G with $d(e,g) \leq r$. That is, $\phi(r)$ is the number of group elements g in the closed metric r–ball centered at e. If G is the free group on k generators, show that

$$\phi(r) = \frac{k(2k-1)^r - 1}{k-1}$$

Try some test cases first to get an idea what's going on. Note that if you toss in the inverses to the generating elements so that S now has $2k$ elements, you can consider words with only positive exponents.

c) If G is the free abelian group on k generators, show that

$$\phi(r) = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{r}{i}$$

Hint: It's combinatorics!

Cultural Note: If you analyze the asymptotic behavior as $r \to \infty$ of $\phi(r)$, you can see that the free group has *exponential growth* while the free abelian group has *polynomial growth* (Check this yourself). Think of $\phi(r)$ as measuring the volume of a metric ball in G. If you compare this to Euclidean n–space (curvature $= 0$), you will notice that metric balls grow like Cr^n, while metric balls in hyperbolic space (curvature < 0) (for those who had Math 404) grow exponentially. It turns out that one can show that the fundamental group (a measure of topological complexity) of a nonnegatively curved manifold can have at most polynomial growth, and that negatively curved manifolds must have a fundamental group that exhibits exponential growth. Thus, there is a link between the growth of metric balls in the manifold and the growth of the fundamental group of the manifold.

Question 2 Let $D(x,r) \subset X$ denote the set of points $\{y \in X \mid d(x,y) \leq r\}$, the closed metric r–ball. Prove that $\overline{B(x,r)} \subset D(x,r)$. Show, by example, that in general, $\overline{B(x,r)} \neq D(x,r)$. Obviously, X will have to be different than $\mathbb{R}^n$.