Math 541

Lecture on 1-manifolds

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Material

- Review of Items from Karen and Tinh's Lecture
  - Statement of Theorem
  - Parametrization by Arc-length
  - Statement of the relevant lemma
  - Possible graphs Γ, the set where two parameterizations of a 1-manifold agree.
- Relationship between the graph of Γ and the parameterized sections of a 1-manifold.
- Extending Γ if there is only one connected component.
- Using Γ to find a diffeomorphism with S^1 if there are two connected components.
- Using the lemma to prove the classification theorem.

Statement of Theorem

Theorem: Any smooth, connected 1-dimensional manifold M is diffeomorphic either to the circle S^1 or to some interval of real numbers.

For the rest of this presentation, M will be a 1-dimensional manifold sitting in n-dimensional Euclidean space with $n \geq 1$.

Because any interval in $\mathbb{R}$ is diffeomorphic to $[0, 1], (0, 1]$, or $(0, 1)$, it follows that there are only 4 distinct 1-manifolds up to diffeomorphism. For the rest of this presentation I and J will refer to intervals in $\mathbb{R}$.

Parametrization by Arc-length

Definition: A map $f : I \to M$ is a *parametrization by arc-length* if f maps I diffeomorphically onto an open subset of M, and if the "velocity vector" $Df_s(1) \in TM_{f(s)}$ has unit length for every $s \in I$.

An easier way to think of this is to recall that for every dimension $n \geq 1$ there is a notion of distance in $\mathbb{R}^n$. If f is a parametrization by arc-length then moving x units in I will result in moving x units along M. In Differential Geometry this is sometimes referred to as a *unit speed parametrization*.

Lemma

Lemma: Let $f : I \to M$ and $g : J \to M$ be parameterizations by arc-length. Then $f(I) \cap g(J)$ has at most two components. If it has one component, then f can be extended to a parametrization by arc-length of the union $f(I) \cup g(J)$. If it has two components then M must be diffeomorphic to S^1.

Proof of Lemma

Proof: As Tinh and Karen showed $g^{-1} \circ f$ is defined only on the parts of I that map into the image $g(J)$.

$$g^{-1} \circ f : f^{-1}(f(I) \cap g(J)) \subset I \to g^{-1}(f(I) \cap g(J)) \subset J$$

Also note that

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle f} \nearrow & & \nwarrow {\scriptstyle g} & \\ I & & \xrightarrow[g^{-1}\circ f]{} & & J \end{array} \quad \text{so} \quad \begin{array}{ccccc} & & T_xM & & \\ & {\scriptstyle Df} \nearrow & & \nwarrow {\scriptstyle Dg} & \\ \mathbb{R} & & \xrightarrow[D(g^{-1}\circ f)]{} & & \mathbb{R} \end{array}$$

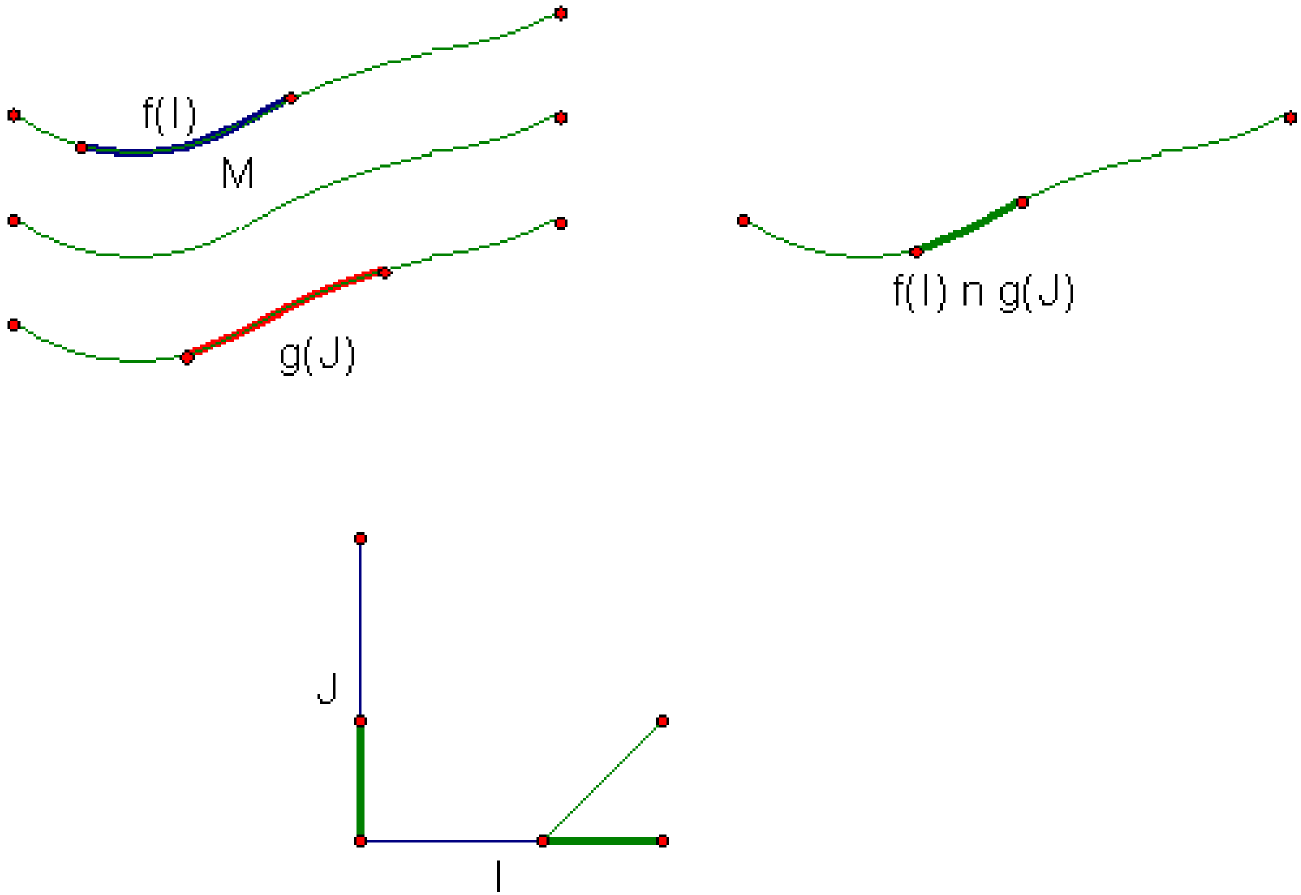
and $D(g^{-1} \circ f) = Dg^{-1} \circ Df = \pm 1 \cdot \pm 1 = \pm 1$.

The graph Γ

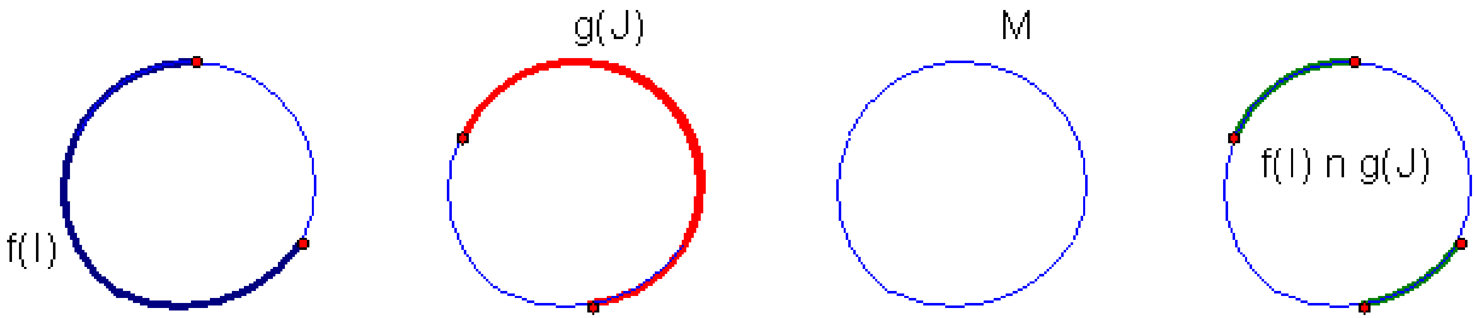
Consider $\Gamma = \{(s,t) | s \in I, t \in J, f(s) = g(t)\}$.
Note that this is the graph of $g^{-1} \circ f$ on $I \times J$.
Tinh and Karen established some facts:

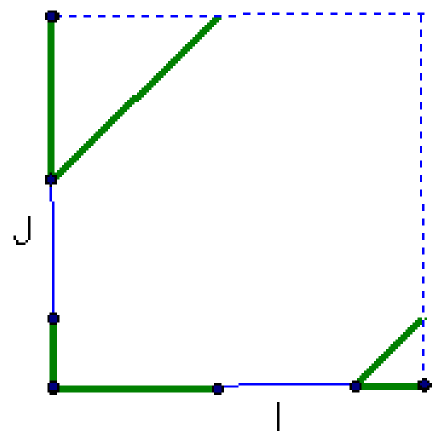
- Γ is closed in $I \times J$.
- Γ must extend to the boundary of $I \times J$.
- Γ has 1 or 2 connected components.
- Γ must have slope ± 1 and both components must have the same slope.
- $g^{-1} \circ f$ is a bijection, so the components of Γ must not overlap when projected onto I or J.

How does Γ relate to 1-manifolds?



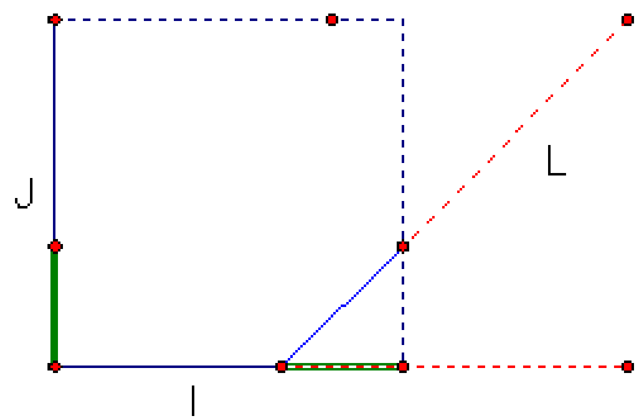
2 connected components?





Extending Γ

If there is only one connected component then there is a way to extend f to have the range $f(I) \cup g(J)$.
Define $L : \mathbb{R} \to \mathbb{R}$ as the extension of $g^{-1} \circ f$.

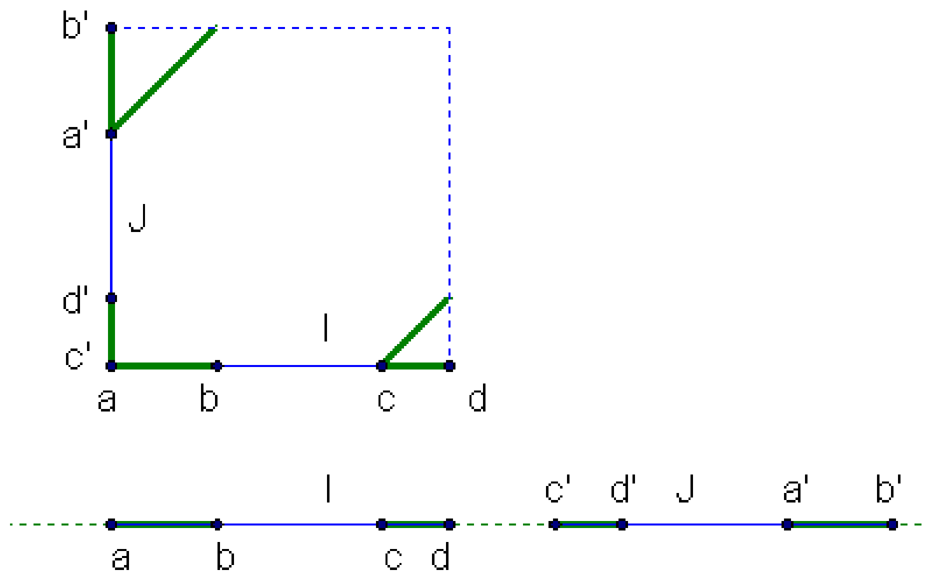


Define $F : I \cup L^{-1}(J) \to f(I) \cup g(J)$ by

$$F(x) = \begin{cases} f(x) & x \in I \\ (g \circ L)(x) & x \in L^{-1}(J) \end{cases}$$

What if Γ has 2 parts?

As both I and J are intervals on the real line it is possible to shift g so that the $c' = c$ and $d' = d$.



We have a nifty demonstration of this in Geometer's Sketchpad!

The diffeomorphism with S^1

Now let t range from a to a' and define

$$\theta = \frac{2\pi(t-a)}{a'-a} \text{ and } t = \frac{(a'-a)\theta}{2\pi} + a$$

This means that θ will range from $\frac{2\pi(a-a)}{a'-a}$ to $\frac{2\pi(a'-a)}{a'-a}$, a difference of

$$\frac{2\pi(a'-a)}{a'-a} - \frac{2\pi(a-a)}{a'-a} = 2\pi$$

so

$$h(\cos\theta, \sin\theta) = \begin{cases} f(t) & t \in I \\ g(t) & t \in J \end{cases}$$

will map all of the S^1 into all of M.

Using the lemma to prove the theorem

Now that the lemma is proven, the proof of the theorem is easy.
Take any parametrization f of M and extend it as far as possible. If it "wraps around" on itself, we are finished. If it does not, then we must show that we have all of M.

Assume that we do not have all of M, then there is a limit point x of $f(I)$ that is in M. We can then take an open neighborhood of x and parameterize it. As x is a limit point of $f(I)$ the neighborhood will have an intersection with $f(I)$ and f can be extended to include M. This is a contradiction as f was assumed to be maximally extended.
This proves the theorem.