Math 440/540 Supplementary Exercises 3 Name:________________________

Question 1 We mentioned in class that $\prod_{\alpha \in A} X_\alpha$ with the product topology is connected if each X_α is connected.
a) Prove this statement for finite products. Hint: Theorem II.10.6 in the textbook.
b) Prove this statement for arbitrary products. Hint: Exercise II.12.6.
c) Show that $\mathbb{R}^\omega$ is disconnected in the box topology. Hint: Write $\mathbb{R}^\omega$ as a union of the collection of bounded sequences and the collection of unbounded sequences.
d) Is $\mathbb{R}^\omega$ connected in the uniform topology?

Question 2 In this problem we will study the real projective 3-space $\mathbb{R}P^3$ a little more. In class we showed that $\mathbb{R}P^3 \cong S^3/\{x \sim -x\} \cong B^3/\{\text{antipodal points of } S^2 = \partial B^3\}$.

Let $SO(3)$ denote the *special orthogonal group of* $\mathbb{R}^3$. That is $SO(3)$ consists of those 3×3 matrices A such that $\det A = 1$ and $A^T A = AA^T = I$. Using elementary linear algebra, one can easily see that $SO(3)$ preserves the standard inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$ on $\mathbb{R}^3$. To wit, $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{x}^T A^T A \vec{y} = (A\vec{x})^T (A\vec{y}) = \langle A\vec{x}, A\vec{y} \rangle$. Thus, A preserves the lengths of vectors and angles between them. Three basis vectors $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{R}^3$ are positively (resp. negatively) oriented if $\det[\vec{x} \mid \vec{y} \mid \vec{z}] > 0$ (resp. < 0). Since $A[\vec{x} \mid \vec{y} \mid \vec{z}] = [A\vec{x} \mid A\vec{y} \mid A\vec{z}]$, we see that A preserves the orientation of basis vectors, since $\det A = 1$. It is a more advanced fact from linear algebra (see, for example, Theorem 14.12 of the Schaum's outline in linear algebra), that for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto A\vec{x}$, where $A \in SO(3)$, there exists an orthonormal basis such that T has a matrix representation relative to this basis which looks like $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$. Thus, T may be regarded as a counterclockwise rotation through angle θ, $(0 \le \theta < 2\pi)$ in a plane orthogonal to a fixed direction.

a) Show that $SO(3) \cong \mathbb{R}P^3$. Hint: Regard B^3 as the solid ball of radius π in $\mathbb{R}^3$. Given $A \in SO(3)$, a rotation through angle θ about a fixed direction $\vec{v}$, identify A with the point $\theta\vec{v}$ of B^3. The key is to figure what to due for rotations with $\theta > \pi$. Such rotations are equivalent to others with $\theta < \pi$.

Note: Since the collection of all 3×3 matrices is linearly homeomorphic to $\mathbb{R}^9$, we have shown that $\mathbb{R}P^3 \cong SO(3)$ can be regarded as a 3-dimensional (non-linear) submanifold of $\mathbb{R}^9$, and that $\mathbb{R}P^3$ inherits the structure of a *topological group* (that is, a group G so that the operation $(g, h) \mapsto gh$ is a continuous map from $G \times G \to G$ and $g \mapsto g^{-1}$ is a continuous map from $G \to G$). In fact, all of this can be done in the smooth category, and thus $\mathbb{R}P^3 \cong SO(3)$ is an example of a *Lie group.*

Now we investigate one more interpretation of $\mathbb{R}P^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere. At each point $p \in S^2$, there is a 2-dimensional tangent plane, denoted T_pS^2. By translating the plane back to the origin, we regard T_pS^2 as a 2-dimensional linear subspace of $\mathbb{R}^3$. The *tangent bundle of* S^2, denoted TS^2 is the collection of all tangent vectors at all points of S^2. That is, $TS^2 = \bigcup_{p \in S^2} T_pS^2 = \{(p, \vec{v}) \mid p \in S^2 \text{ and } \vec{v} \in T_pS^2\}$. TS^2 can be given the structure of a 4-dimensional manifold (it is an example of a non-trivial *vector bundle*: $TS^2 \ncong S^2 \times \mathbb{R}^2$). We may also look at the *unit tangent bundle* T_1S^2: $T_1S^2 = \{(p, \vec{v}) \mid (p, \vec{v}) \in TS^2 \text{ and } \|\vec{v}\| = 1\}$.

b) Show that $T_1S^2 \cong \mathbb{R}P^3$ by showing that $T_1S^2 \cong SO(3)$. Hint: Let $\vec{v}_2$ be a unit tangent vector to S^2 based at p. Then if we regard the point p itself as a unit vector $\vec{v}_1$, the tangent plane at p is spanned by $\vec{v}_2$ and $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. Thus, we have a bijective correspondence between unit tangent vectors to S^2 and orthonormal triples: $\vec{v}_2 \leftrightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. There are just a few more steps - complete the proof and fill in the gaps.
Application: The homeomorphism between $\mathbb{R}P^3$ and T_1S^2 can be used to show that if your head is a (smooth) 2-sphere, then you can't comb your hair without a part.

Summary: We now have five different interpretations of $\mathbb{R}P^3$:

1.	$S^3/\{x \sim -x\}$	2.	$B^3/\{\text{antipodal points of } S^2 = \partial B^3\}$
3.	$SO(3)$	4.	T_1S^2, the unit tangent bundle to S^2
5.	The collection of 1-dimensional subspaces of $\mathbb{R}^4$		