Math 440/540 Supplementary Exercises 1 Name:________________________

Question 1 (Introduction to geometric group theory) Let G be a finitely generated group. Let $S = \{g_1, g_2, \ldots, g_k\}$ be a finite system of generators. Thus any element $g \in G$ can be written a product $g = g_{k_1}^{r_1} g_{k_2}^{r_2} \cdots g_{k_\ell}^{r_\ell}$, where $r_i \in \mathbb{Z}$ of elements in the generating set S. Define a pre-norm on G, by $|||g||| = \sum |r_i|$. Since an element $g \in G$ can have multiple representations as a product of elements of the generating set, we define a norm on G by

$$\|g\| = \min_{\text{representations of } g} |||g|||$$

Note that if $e \in G$ is the identity element, then $\|e\| = 0$. Define a metric on G by setting $d(g, h) = \|gh^{-1}\|$.

a) Show that this defines a metric on G (that depends on generating set S.) This metric is called the *word metric* on G, and it turns out that study of G as a metric space can lead to algebraic information about the group G.
b) Let $\phi(r)$ denote the number of elements in G with $d(e, g) \leq r$. That is, $\phi(r)$ is the number of group elements g in the closed metric r–ball centered at e. If G is the free group on k generators, show that

$$\phi(r) = \frac{k(2k-1)^r - 1}{k-1}$$

Try some test cases first to get an idea what's going on. Note that if you toss in the inverses to the generating elements so that S now has $2k$ elements, you can consider words with only positive exponents.
c) If G is the free abelian group on k generators, show that

$$\phi(r) = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{r}{i}$$

Cultural Note: If you analyze the asymptotic behavior as $r \to \infty$ of $\phi(r)$, you can see that the free group has *exponential growth* while the free abelian group has *polynomial growth* (Check this yourself). Think of $\phi(r)$ as measuring the volume of a metric ball in G. If you compare this to Euclidean n–space (curvature $= 0$), you will notice that the n-dimensional volume of metric balls grow polynomially ($\sim Cr^n$), while volumes of metric balls in hyperbolic space (curvature < 0) (for those who had Math 404) grow exponentially. It turns out that one can show that the fundamental group (a measure of topological complexity) of a nonnegatively curved manifold can have at most polynomial growth, and that negatively curved manifolds must have a fundamental group that exhibits exponential growth. Thus, there is a link between the volume growth of metric balls in the manifold and the growth of the fundamental group of the manifold as defined above.

Question 2 This is an enhancement of exercise 1.1.12-13 in the text. Let (X, d) be a metric space. The collection $\mathcal{T}$ of open sets of X is called *the metric topology on X relative to metric d*. A different metric d' on X gives rise to another metric topology $\mathcal{T}'$. The topology $\mathcal{T}'$ is said to be *finer* than the topology $\mathcal{T}$ if $\mathcal{T}' \supset \mathcal{T}$. The topology $\mathcal{T}'$ is said to be *coarser* than the topology $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$. The topologies are *equivalent* if $\mathcal{T}' = \mathcal{T}$.

a) Show that the function $\overline{d}(x, y) = \min\{1, d(x, y)\}$ is a metric. $\overline{d}$ is called the *standard bounded metric* corresponding to d.
b) Show that the function $d'(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ is a metric. This metric is also bounded. Hint: Use the mean value theorem on the function $f(x) = \dfrac{x}{1 + x}$ for $x \geq 0$ to help verify the triangle inequality.
c) Show that two metric topologies on X are equivalent if and only if they have the same convergent sequences.
d) Show that the metric topologies (X, d), $(X, \overline{d})$, and (X, d') are all equivalent. Thus, all metric spaces are equivalent to a bounded metric space.
e) Let d and ρ be two metrics on a set X. Suppose that there exists constants $\alpha, \beta > 0$ such that $\alpha d(x, y) \leq \rho(x, y) \leq \beta d(x, y)$ for all $x, y \in X$. Show that the two resulting metric topologies are equivalent. Is this condition necessary for the topologies to be equivalent?
f) Let d and ρ be two metrics on a set X. Show that the ρ-metric topology $\mathcal{T}_\rho$ is finer than the d-metric topology $\mathcal{T}_d$ if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_\rho(x, \delta) \subset B_d(x, \varepsilon)$.