

Question 1 (This material is related to material found in section 2.10-12 of the text) Recall that a basis for a topology \mathcal{T} on a set X is a subcollection $\mathcal{B} \subset \mathcal{T}$ of open sets such that any open set $U \subset X$ is a union of the elements of \mathcal{B} . Consider the set $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$, the countable infinite cartesian product of \mathbb{R} with itself. Here, \mathbb{R} is given its usual topology. We define three topologies on \mathbb{R}^ω . The first is the *box topology* \mathcal{T}_b , for which basis elements consist of sets of the form $\prod U_i$, where U_i is open in \mathbb{R} for each $i \in \mathbb{N}$. The second, and most useful, is the *product topology*, \mathcal{T}_p , for which basis elements consist of sets of the form $\prod U_i$, where U_i is open in \mathbb{R} for each $i \in \mathbb{N}$ and $U_i = \mathbb{R}$ for all but finitely many i . The third is the *uniform topology*, \mathcal{T}_u , which is a metric topology defined by the metric $\bar{\rho}(x, y) = \sup \{\bar{d}(x_i, y_i)\}$, where \bar{d} is the standard bounded metric on \mathbb{R} defined in supplementary exercises 1, question 2 with $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots) \in \mathbb{R}^\omega$.

a) Show that the box topology is finer than the uniform topology which is in turn finer than the product topology. That is, $\mathcal{T}_b \supset \mathcal{T}_u \supset \mathcal{T}_p$.

b) Explain why $\mathcal{T}_u \supset \mathcal{T}_p$ does not immediately imply that \mathcal{T}_p is metrizable. That is, is it true that a topology which is coarser than a metric topology must be metrizable? It turns out that the product topology is metrizable if one uses the metric $D(x, y) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$. (You do not need to verify this: checking that it is a metric is easy, that it induces the product topology is not hard, but not obvious either).

The next four problems investigate differences between the box, uniform, and product topologies on \mathbb{R}^ω .

c) Consider the following three functions from $\mathbb{R} \rightarrow \mathbb{R}^\omega$: $f(t) = (t, 2t, 3t, \dots)$, $g(t) = (t, t, t, \dots)$, $h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$. Which are continuous assuming \mathbb{R}^ω is given the box, uniform or product topologies? Hint: Since the uniform and product topologies are metrizable, you can actually compute the (uniform or product) distance between $f(x)$ and $f(y)$ for example and compare it to $|x - y| = d(x, y)$.

d) In which topologies do the following sequences converge?

$$\begin{array}{llll} w_1 = (1, 1, 1, 1, \dots), & x_1 = (1, 1, 1, 1, \dots) & y_1 = (1, 0, 0, 0, \dots), & z_1 = (1, 1, 0, 0, \dots), \\ w_2 = (0, 2, 2, 2, \dots), & x_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) & y_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots), & z_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots), \\ w_3 = (0, 0, 3, 3, \dots), & x_3 = (0, 0, \frac{1}{3}, \frac{1}{3}, \dots) & y_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots), & z_3 = (\frac{1}{3}, \frac{1}{3}, 0, 0, \dots), \\ \dots & \dots & \dots & \dots \end{array}$$

e) Let $0 = (0, 0, 0, \dots) \in \mathbb{R}^\omega$ and let $0 < \varepsilon < 1$. Define $U(0, \varepsilon) = (-\varepsilon, \varepsilon) \times \cdots \times (-\varepsilon, \varepsilon) \times \cdots$. Show that $U(0, \varepsilon)$ is not equal to the uniform ε -ball $B_{\bar{\rho}}(0, \varepsilon)$ and that $U(0, \varepsilon)$ is not open in the uniform topology. Finally, show that $B_{\bar{\rho}}(0, \varepsilon) = \bigcup_{\delta < \varepsilon} U(0, \delta)$.

f) Let X be a metric space. Recall that if $Y \subset X$, then if $x \in \bar{Y}$, then there exists a sequence $\{y_n\} \subset Y$ with $y_n \rightarrow x$ (Theorem 1.1.11 in text). We use this to show that $X = \mathbb{R}^\omega$ with the box topology is not metrizable. Let $Y = \{(x_1, x_2, \dots) \mid x_i > 0 \text{ for all } i \in \mathbb{N}\}$. First, show that $0 = (0, 0, 0, \dots) \in \bar{Y}$. Now show that there is no sequence in Y that can converge to 0. Hint: Suppose such a sequence exists and find a box topology neighborhood of 0 that does not contain any element of the sequence.

g) Let J be any uncountable set. Show that $\mathbb{R}^J = \prod_{\alpha \in J} \mathbb{R}$ (the uncountable cartesian product of \mathbb{R} with itself) is not metrizable in the product topology. To do this, consider the subset $Y = \{(x_\alpha) \in \mathbb{R}^J \mid x_\alpha = 1 \text{ for all but finitely many } \alpha\}$. Show that $0 = (0_\alpha) \in \bar{Y}$. That is, show that any product topology basis element $\prod U_\alpha$ containing 0 has nontrivial intersection with Y . Now let $\{y_n\}_{i=1}^\infty \subset Y$ be any sequence. Show that this sequence cannot converge to 0. Thus, as above, we conclude that \mathbb{R}^J is not metrizable (in the product topology).