A MORE RIGOROUS APPROACH TO THE DIVERGENCE OF A VECTOR FIELD

JOSEPH E. BORZELLINO

1. Introduction

We prove the following theorem on the divergence of a vector field.

Theorem 1.1. *Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ be a system of differential equations in $\mathbb{R}^n$ with flow $\phi(t;\mathbf{x})$. Let $\mathcal{D}$ be a region in $\mathbb{R}^n$ with finite n–dimensional volume and piecewise smooth boundary $\partial\mathcal{D}$. Let $\mathcal{D}(t)$ be the region formed by flowing along for time t,*

$$\mathcal{D}(t) = \{\phi(t;\mathbf{x}) \mid \mathbf{x} \in \mathcal{D}\}.$$

Let $V(t)$ be the n–dimensional volume of $\mathcal{D}(t)$. Then

$$\frac{d}{dt}V(t) = \int_{\mathcal{D}(t)} \operatorname{div}\mathbf{F}(\mathbf{x})\, dV$$

Much of what follows is adapted from [3].

2. Preliminary Results

The first result is about differentiating determinants.

Lemma 2.1. *Let $A(t) = [\mathbf{v}_1(t) \mid \cdots \mid \mathbf{v}_n(t)]$ be an $n \times n$ matrix with columns $\mathbf{v}_i(t)$ which depend on t. Then*

$$\begin{aligned}\frac{d}{dt}\det A(t) = &\det\left[\frac{d}{dt}\mathbf{v}_1(t) \mid \cdots \mid \mathbf{v}_n(t)\right] + \det\left[\mathbf{v}_1(t) \mid \frac{d}{dt}\mathbf{v}_2(t) \mid \cdots \mid \mathbf{v}_n(t)\right] \\ &+ \cdots + \det\left[\mathbf{v}_1(t) \mid \mathbf{v}_2(t) \mid \cdots \mid \frac{d}{dt}\mathbf{v}_n(t)\right]\end{aligned}$$

Proof. The proof follows trivially from the multilinearity of the determinant as a function of its columns. We have the identity:

$$\begin{aligned}\det A(t+h) = &\det[\mathbf{v}_1(t+h) \mid \cdots \mid \mathbf{v}_n(t+h)] = \\ &\det[\mathbf{v}_1(t+h) - \mathbf{v}_1(t) \mid \mathbf{v}_2(t+h) \mid \cdots \mid \mathbf{v}_n(t+h)] \\ &+ \det[\mathbf{v}_1(t) \mid \mathbf{v}_2(t+h) - \mathbf{v}_2(t) \mid \cdots \mid \mathbf{v}_n(t+h)] \\ &+ \cdots \\ &+ \det[\mathbf{v}_1(t) \mid \mathbf{v}_2(t) \mid \cdots \mid \mathbf{v}_n(t+h) - \mathbf{v}_n(t)] \\ &+ \det[\mathbf{v}_1(t) \mid \mathbf{v}_2(t) \mid \cdots \mid \mathbf{v}_n(t)]\end{aligned}$$

Thus, $\dfrac{\det A(t+h) - \det A(t)}{h} =$

$$\begin{aligned}
&\det\left[\frac{\mathbf{v}_1(t+h)-\mathbf{v}_1(t)}{h} \mid \mathbf{v}_2(t+h) \mid \cdots \mid \mathbf{v}_n(t+h)\right] \\
&+ \det\left[\mathbf{v}_1(t) \mid \frac{\mathbf{v}_2(t+h)-\mathbf{v}_2(t)}{h} \mid \cdots \mid \mathbf{v}_n(t+h)\right] \\
&+ \cdots \\
&+ \det\left[\mathbf{v}_1(t) \mid \mathbf{v}_2(t) \mid \cdots \mid \frac{\mathbf{v}_n(t+h)-\mathbf{v}_n(t)}{h}\right]
\end{aligned}$$

The proof follows by taking a limit as $h \to 0$. □

The next lemma is known as the Liouville formula or Abel's formula.

Lemma 2.2. *Let $M(t)$ be a fundamental matrix solution for the linear system of differential equations $\dot{\mathbf{x}} = A(t)\mathbf{x}$. That is, $M'(t) = A(t)M(t)$ and $M(t_0)$ is invertible for some $t_0 \in \mathbb{R}$. Then*

$$\frac{d}{dt}\det M(t) = \operatorname{trace}(A(t))\det M(t)$$

Proof. Let $\{\mathbf{e}_i\}$ denote the standard basis for $\mathbb{R}^n$. For $t_0 \in \mathbb{R}$ we have

$$\det\left(M(t)M(t_0)^{-1}\right) = \det\left[M(t)M(t_0)^{-1}\mathbf{e}_1 \mid \cdots \mid M(t)M(t_0)^{-1}\mathbf{e}_n\right].$$

By lemma 2.1,

$$\begin{gathered}
\frac{d}{dt}\Big|_{t=t_0}\det\left(M(t)M(t_0)^{-1}\right) = \\
\det\left[\frac{d}{dt}\Big|_{t=t_0}M(t)M(t_0)^{-1}\mathbf{e}_1 \mid \cdots \mid M(t_0)M(t_0)^{-1}\mathbf{e}_n\right] + \\
\det\left[M(t_0)M(t_0)^{-1}\mathbf{e}_1 \mid \frac{d}{dt}\Big|_{t=t_0}M(t)M(t_0)^{-1}\mathbf{e}_2 \mid \cdots \mid M(t_0)M(t_0)^{-1}\mathbf{e}_n\right] + \cdots + \\
\det\left[M(t_0)M(t_0)^{-1}\mathbf{e}_1 \mid M(t_0)M(t_0)^{-1}\mathbf{e}_2 \mid \cdots \mid \frac{d}{dt}\Big|_{t=t_0}M(t)M(t_0)^{-1}\mathbf{e}_n\right] = \\
\det\left[A(t_0)M(t_0)M(t_0)^{-1}\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \det\left[\mathbf{e}_1 \mid A(t_0)M(t_0)M(t_0)^{-1}\mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \cdots + \\
\det\left[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid A(t_0)M(t_0)M(t_0)^{-1}\mathbf{e}_n\right] = \\
\det\left[A(t_0)\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \det\left[\mathbf{e}_1 \mid A(t_0)\mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \cdots + \\
\det\left[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid A(t_0)\mathbf{e}_n\right] = \\
\det\left[\sum_{j=1}^{n} a_{j1}(t_0)\mathbf{e}_j \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \cdots + \det\left[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \sum_{j=1}^{n} a_{jn}(t_0)\mathbf{e}_j\right] = \\
\det\left[a_{11}(t_0)\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\right] + \cdots + \det\left[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid a_{nn}(t_0)\mathbf{e}_n\right] \\
= \sum_{i=1}^{n} a_{ii}(t_0) = \operatorname{trace}(A(t_0))
\end{gathered}$$

Using the fact that the determinant of a product is the product of the determinants, the first and last lines above can be rewritten as $\frac{d}{dt}|_{t=t_0} \det M(t) = \operatorname{trace}(A(t_0)) \det M(t_0)$. □

The next lemma is often referred to as the first variation formula.

Lemma 2.3. *For a C^1 vector field $\mathbf{F}$, the flow $\phi(t;\mathbf{x}_0)$ of the system of differential equations $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ satisfies*

$$\frac{d}{dt} D_{\mathbf{x}}\phi(t;\mathbf{x}_0) = D\mathbf{F}(\phi(t;\mathbf{x}_0)) D_{\mathbf{x}}\phi(t;\mathbf{x}_0).$$

where $D_{\mathbf{x}}\phi(t;\mathbf{x}_0) = \left[\dfrac{\partial \phi_i}{\partial x_j}(t;\mathbf{x}_0)\right]$ is the matrix of partial derivatives with respect to $\mathbb{R}^n$ and $D\mathbf{F}(\phi(t;\mathbf{x}_0)) = \left[\dfrac{\partial F_i}{\partial x_j}(\phi(t;\mathbf{x}_0))\right]$.

Proof. From the theory of dynamical systems all required derivatives exist and interchange of order of differentiation is justified [1]. Thus,

$$\frac{d}{dt} D_{\mathbf{x}}\phi(t;\mathbf{x}) = D_{\mathbf{x}} \frac{d}{dt}\phi(t;\mathbf{x}) = D_{\mathbf{x}}(\mathbf{F}(\phi(t;\mathbf{x})) = D\mathbf{F}(\phi(t;\mathbf{x})) D_{\mathbf{x}}\phi(t;\mathbf{x})$$

where the last equality is just the chain rule. □

3. Proof of Theorem 1.1

Since $\phi(0;\mathbf{x}) = \mathbf{x}$, we have $D_{\mathbf{x}}\phi(0;\mathbf{x}) = I$, the identity map. Since $\phi(t;\mathbf{x})$ is a diffeomorphism we have that $\det D_{\mathbf{x}}\phi(t;\mathbf{x}) > 0$ for all $t \geq 0$ for which the flow is defined. Thus, by the change of variables formula for multiple integration we have:

$$V(t) = \int_{\mathcal{D}(t)} 1 \, dV = \int_{\mathcal{D}} \det D_{\mathbf{x}}\phi(t;\mathbf{x}) \, dV$$

Hence,

$$\frac{d}{dt} V(t) = \int_{\mathcal{D}} \frac{d}{dt} \det D_{\mathbf{x}}\phi(t;\mathbf{x}) \, dV = \int_{\mathcal{D}} \operatorname{trace}(D\mathbf{F}(\phi(t;\mathbf{x})) \det D_{\mathbf{x}}\phi(t;\mathbf{x}) \, dV$$

$$= \int_{\mathcal{D}} \operatorname{div} \mathbf{F}(\phi(t;\mathbf{x})) \det D_{\mathbf{x}}\phi(t;\mathbf{x}) \, dV = \int_{\mathcal{D}(t)} \operatorname{div} \mathbf{F}(\mathbf{x}) \, dV$$

Lemma 2.3 implies that the matrix $D_{\mathbf{x}}\phi(t;\mathbf{x})$ is a fundamental matrix solution to a system of differential equations. Thus, the third equality above follows from lemma 2.2 applied to the matrix $D_{\mathbf{x}}\phi(t;\mathbf{x})$. The last equality is the change of variables formula again.

Corollary 3.1. $\dfrac{d}{dt} V(t) = \displaystyle\int_{\partial\mathcal{D}(t)} (\mathbf{F}\cdot\mathbf{N}) \, dS$, *where $\mathbf{N}$ denotes the outward pointing unit normal to $\partial\mathcal{D}(t)$.*

Proof. Note that since $\partial\mathcal{D}(t)$ is $(n-1)$-dimensional, the integral in the corollary is an $(n-1)$-dimensional integral over a piecewise smooth hypersurface in $\mathbb{R}^n$. In particular, in dimension 2, this follows by applying the divergence form of Green's theorem to the result of theorem 1.1. Likewise in dimension 3, the result follows from the classical Gauss divergence theorem. In higher dimensions, the result follows from the generalized divergence theorem [2]. □

4. Appendix: Integration over Hypersurfaces in $\mathbb{R}^n$

In this appendix, we give provide the background necessary to compute the integral that appears in corollary 3.1. Much of what follows is adapted from [2]. Let $M^{n-1} \subset \mathbb{R}^n$ be an $(n-1)$-dimensional orientable hypersurface given by a parametrization $\Phi : D \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$.

$$\Phi(u_1, \ldots, u_{n-1}) = (f_1(u_1, \ldots, u_{n-1}), \ldots, f_n(u_1, \ldots, u_{n-1}))$$

We assume that the columns Φ_{u_i} of $D\Phi$ are linearly independent at each point $u \in D$, and thus they span T_pM, the tangent space of M at $p = \Phi(u)$. Since each tangent space is $(n-1)$-dimensional, there is a well-defined (up to choice of $\pm$) smooth unit normal vector field $\mathbf{N}$ on M. For concreteness, we choose $\mathbf{N}$ so that the matrix

$$A = \begin{bmatrix} \mathbf{N} & \Phi_{u_1} & \ldots & \Phi_{u_{n-1}} \end{bmatrix}$$

has $\det A > 0$.

How can we compute a unit normal? In $\mathbb{R}^3$, $\mathbf{N}$ is easy to compute since it must be parallel to the cross product $\Phi_{u_1} \times \Phi_{u_2}$. We need an analogue of the cross product in $\mathbb{R}^n$. We can do this with the help of a proposition:

Proposition 4.1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ be linearly independent vectors in $\mathbb{R}^n$ and let X be the $n \times (n-1)$ matrix that has $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ as its columns. Let $\mathbf{c} = (c_1, \ldots, c_n)$ be the vector in $\mathbb{R}^n$ where*

$$c_i = (-1)^{i-1} \det X(1, \ldots, \hat{i}, \ldots, n)$$

Here $X(1, \ldots, \hat{i}, \ldots, n)$ is the $(n-1) \times (n-1)$matrix obtained from X by deleting the i-th row. Then $\mathbf{c}$ has the following properties:

1. *$\mathbf{c}$ is non-zero and orthogonal to each $\mathbf{x}_i$.*
2. *$\det \begin{bmatrix} \mathbf{c} & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix} > 0$.*
3. *$\|\mathbf{c}\|$ = volume of parallelepiped spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} = \mathrm{Vol}(X)$.*

Note that when $n = 3$, the vector $\mathbf{c} = \mathbf{x}_1 \times \mathbf{x}_2$ and thus $\mathbf{c}$ is referred to as the generalization of cross product to $\mathbb{R}^n$. It is often denoted by $\mathbf{x}_1 \times \cdots \times \mathbf{x}_{n-1}$.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be any vector. Then by expanding the determinant down the first column we have

$$\det \begin{bmatrix} \mathbf{a} & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix} = \sum_{i=1}^{n} a_i (-1)^{i-1} \det X(1, \ldots, \hat{i}, \ldots, n) = \mathbf{a} \cdot \mathbf{c}$$

Thus, if $\mathbf{a} = \mathbf{x}_i$, we have, $\mathbf{x}_i \cdot \mathbf{c} = \det \begin{bmatrix} \mathbf{x}_i & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix} = 0$ since the matrix has two identical columns. Hence $\mathbf{c}$ is orthogonal to each $\mathbf{x}_i$.

To see that $\mathbf{c}$ is non-zero, observe that columns of X span an $(n-1)$-dimensional subspace and so $\mathrm{rank}(X) = (n-1)$. A basic theorem of linear algebra states that $\mathrm{rank}(X) = \mathrm{rank}(X^T)$ and thus the rows of X span a subspace of dimension $(n-1)$ as well. If row i is the linearly dependent row, then $\det X(1, \ldots, \hat{i}, \ldots, n) \neq 0$ whence $c_i \neq 0$ and thus $\mathbf{c}$ is non-zero. This proves (1). If $\mathbf{a} = \mathbf{c}$, then $\det \begin{bmatrix} \mathbf{c} & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix} = \mathbf{c} \cdot \mathbf{c} = \|\mathbf{c}\|^2 > 0$. This proves (2). To prove (3), note that

$$\begin{bmatrix} \mathbf{c} & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix}^T \begin{bmatrix} \mathbf{c} & \mathbf{x}_1 & \ldots & \mathbf{x}_{n-1} \end{bmatrix} = \begin{bmatrix} \|\mathbf{c}\|^2 & 0 \\ 0 & X^T X \end{bmatrix}$$

By taking determinant of both sides yields

$$\|\mathbf{c}\|^4 = \det\begin{bmatrix}\mathbf{c} & \mathbf{x}_1 & \dots & \mathbf{x}_{n-1}\end{bmatrix}^T \det\begin{bmatrix}\mathbf{c} & \mathbf{x}_1 & \dots & \mathbf{x}_{n-1}\end{bmatrix} = \|\mathbf{c}\|^2 \operatorname{Vol}(X)^2$$

from which it follows that $\|\mathbf{c}\| = \operatorname{Vol}(X)$. Note that we used the fact that $\operatorname{Vol}(X) = \sqrt{\det(X^T X)}$. □

Now we return to our discussion of surface integrals. By proposition 4.1, if we let $X = D\Phi$, then $\mathbf{N} = \mathbf{c}/\|\mathbf{c}\|$ is a unit normal vector field over M. Let $\mathbf{F}$ be a continuous vector field defined over M. Then we may define the surface integral of $\mathbf{F}$ over M to be:

$$\begin{aligned}\int_M \mathbf{F}\cdot d\mathbf{S} &= \int_M (\mathbf{F}\cdot\mathbf{N})\, dS \\ &= \int_D (\mathbf{F}\cdot\mathbf{N})\sqrt{\det\left[(D\Phi)^T(D\Phi)\right]}\; dx_1\dots dx_{n-1} \\ &= \int_D (\mathbf{F}\cdot\mathbf{c})\frac{\sqrt{\det\left[(D\Phi)^T(D\Phi)\right]}}{\|\mathbf{c}\|}\; dx_1\dots dx_{n-1} \\ &= \int_D (\mathbf{F}\cdot\mathbf{c})\; dx_1\dots dx_{n-1} \\ &= \int_D (\mathbf{F}\cdot\Phi_{u_1}\times\cdots\times\Phi_{u_{n-1}})\; dx_1\dots dx_{n-1}\end{aligned}$$

This agrees with the classical formula in $\mathbb{R}^3$.

References

[1] Morris W. Hirsch and Stephen Smale, *Differential equations, dynamical systems, and linear algebra*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, Pure and Applied Mathematics, Vol. 60. MR 0486784 (58 #6484)

[2] James R. Munkres, *Analysis on manifolds*, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1991. MR 1079066 (92d:58001)

[3] R. Clark Robinson, *An introduction to dynamical systems: continuous and discrete*, Pearson Prentice Hall, Upper Saddle River, NJ, 2004. MR 2128037 (2005k:37001)

Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA 93407 USA

E-mail address: jborzell@calpoly.edu