Orbifolds of Maximal Diameter

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ABSTRACT. In this paper the Maximal Diameter Theorem of Riemannian geometry is proven for Riemannian orbifolds. In particular, it is shown that a complete Riemannian orbifold with Ricci curvature bounded below by $(n-1)$ and diameter $= \pi$, must have constant sectional curvature 1, and must be a quotient of the sphere (S^n, can) of constant sectional curvature 1 by a subgroup of the orthogonal group $\mathbb{O}(n+1)$ acting discontinuously and isometrically on S^n. It is also shown that the singular locus of the orbifold forms a geometric barrier to the length minimization property of geodesics. We also extend the Bishop relative volume comparison theorem to Riemannian orbifolds.

Introduction. In this paper we wish to examine a generalization of the maximal diameter theorem of Cheng to Riemannian orbifolds. Roughly speaking, a Riemannian orbifold is a metric space locally modelled on quotients of Riemannian manifolds by finite groups of isometries. The term *orbifold* was coined by W. Thurston sometime around the year 1976–77. The term is meant to suggest the orbit space of a group action on a manifold. A similar concept was introduced by I. Satake in 1956, where he used the term *V–manifold* (See [S1]). The "V" was meant to suggest a cone–like singularity. Since then, orbifold has become the preferred terminology. Riemannian orbifolds have recently come up in the study of convergence of Riemannian manifolds. For instance, Anderson [A] has shown that metric spaces in the Gromov–Hausdorff closure of the set of compact Riemannian manifolds with bounded Ricci curvature, lower volume bound, uniform upper diameter bound, and upper bound on the $L^{n/2}$–norm of the Ricci tensor have the structure of an orbifold with only a finite number of singular points. Also, Fukaya [F] has shown that if a sequence of manifolds $\{M_i\}$ in the class of compact n–dimensional Riemannian manifolds with bounded sectional curvature, upper diameter bound, converge to a metric space X with $\dim(X) = n-1$, then X has the structure of a Riemannian orbifold. With these results in mind it seems reasonable that one should try to understand clearly the geometry of orbifolds. Orbifolds are a nice class of singu-

Indiana University Mathematics Journal ©, Vol. 42, No. 1 (1993)

lar spaces because they have a well–defined local structure. In fact, Riemannian orbifolds are Riemannian manifolds except for a nowhere dense set of *singular* points.

Recall that if M is a complete connected n–dimensional Riemannian manifold with $\mathrm{Ric}(M) \geq (n-1)$, then Myers' theorem (see [M]) implies that the diameter $\mathrm{diam}(M) \leq \pi$. In the case that $\mathrm{diam}(M) = \pi$, Cheng's theorem (see [C]) says that $M = S^n$ where S^n is the sphere of constant sectional curvature 1. We will show that Myers' theorem holds for Riemannian orbifolds, and investigate those orbifolds with maximal diameter. In particular we prove the following two results:

Theorem 1. *Let O be a complete n–dimensional Riemannian orbifold with* $\mathrm{Ric}_O \geq (n-1)$ *and* $\mathrm{diam}(O) = \pi$. *Then O is a good orbifold.*

Theorem 2. *Let $O = (M, \Gamma)$ be a complete good n–dimensional Riemannian orbifold with* $\mathrm{Ric}_M \geq (n-1)$. *If* $\mathrm{diam}(O) = \pi$ *then $M = S^n$ and $O = S^n/\Gamma$, where $\Gamma \subset \mathbb{O}(n+1)$ is a finite group of isometries of $\mathbb{R}^{n+1}$. Furthermore, either $O = S^n$ or O is a closed hemisphere, or $O = \Sigma_{\sin}^{n-m} X$, for some $1 \leq m < n$, where $X = S^m/\Gamma$ with* $\mathrm{diam}(X) \leq \frac{1}{2}\pi$.

Example 3. Consider the following singular space: Let $X = \Sigma_{\sin} S^2(\frac{1}{2})$, the sin–suspension of $S^2(\frac{1}{2})$, where $S^2(\frac{1}{2})$ is the sphere of radius $\frac{1}{2}$ in $\mathbb{R}^3$. For now, this suspension can be regarded as the two point compactification of the warped product $(0, \pi) \times_{\sin} S^2(\frac{1}{2})$. A more general definition will be given later. Then the Toponogov curvature of X is ≥ 1, and the diameter of X is π. In light of the previous theorems, X is *not* an orbifold. It is, however, the quotient of S^4 by a Lie group.

One step in proving Theorem 1, will be to show that the excess (see [GP1]) of O is zero. Excess zero, however is not enough to guarantee that the orbifold O is good as the next example shows.

Example 4. Consider the $\mathbb{Z}_p$–teardrop. This is a Riemannian orbifold whose underlying space is S^2, with a single conical singularity of order p. Locally, this singular point is isometric to a small open neighborhood of the north pole in (S^2, can), modulo a rotation of angle $2\pi/p$ about the z–axis in $\mathbb{R}^3$. This space clearly admits a metric of excess zero, but is not good. See [T].

Note that these two theorems together imply that Riemannian orbifolds of maximal diameter admit a geometric structure (a metric of constant curvature). They are all quotients of (S^n, can) by a finite group of isometries. In fact, Thurston has shown in [T] that an orbifold is good if it admits a geometric structure. We do not, however, use this result. For some related results see [HT].

To prove these results we will need several results about orbifolds. All of these results can be found in the author's Ph.D. thesis [B]. A basic reference on general orbifolds is [T]. It will be shown that Riemannian orbifolds inherit a natural stratified length space structure. In addition, we will prove that minimizing segments in orbifolds cannot pass through the singular set, and that the Bishop relative volume comparison theorem holds for Riemannian orbifolds. As a trivial corollary to the volume comparison theorem we will deduce Myers' theorem for Riemannian orbifolds.

I would like to thank my advisor Professor Peter Petersen for many helpful discussions regarding this work.

Basic Definitions. Following Thurston [T], (see also [S1]), the formal definition of (topological) orbifold is as follows:

Definition 5. *A (topological)* orbifold *O consists of a Hausdorff space X_O called the* underlying space *together with the following additional structure. We assume X_O has a countable basis of open charts U_i which is closed under finite intersections. To each U_i is associated a finite group Γ_i, an effective action of Γ_i on some open subset $\tilde{U}_i$ of $\mathbb{R}^n$, and a homeomorphism $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$. Whenever $U_i \subset U_j$, there is to be an injective homomorphism*

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$$

equivariant with respect to f_{ij} (that is, for $\gamma \in \Gamma_i$, $\tilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\varphi}_{ij}(x)$), such that the diagram below commutes:

$$\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{\varphi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i} & \tilde{U}_j/f_{ij}(\Gamma_i) \\
\uparrow & & \downarrow \\
\varphi_i \Big| & & \tilde{U}_j/\Gamma_j \\
 & & \uparrow \varphi_j \\
U_i & \xrightarrow{\subset} & U_j .
\end{array}$$

□

$\tilde{\varphi}_{ij}$ is to be regarded as being defined only up to composition with elements of Γ_j, and f_{ij} are defined only up to conjugation by elements of Γ_j. In general, it is not true that $\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should be an element $\gamma \in \Gamma_k$ such that $\gamma\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$. Just as in the case for manifolds, the covering $\{U_i\}$ is not an intrinsic part of the structure of an orbifold. We regard two coverings to give the same orbifold structure if they can be combined to give a larger covering still satisfying the definitions. Hence, when we speak of an orbifold, we are speaking of an orbifold with such a maximal cover.

Definition 6. *A* Riemannian *orbifold is obtained as above where we require that the $\tilde{U}_i$ are strongly convex, open (possibly non–complete) Riemannian manifolds diffeomorphic to $\mathbb{R}^n$, the Γ_i are finite groups of isometries acting effectively on $\tilde{U}_i$, and the maps $\tilde{\varphi}_i$ are isometries. Recall that for a Riemannian manifold to be strongly convex means that there exists a unique minimal geodesic joining any two points.*

We will need to distinguish between two types of Riemannian orbifolds.

Definition 7. *A* good Riemannian orbifold *is a pair (M, Γ) where M is a Riemannian manifold and Γ is a (proper) discontinuous group of isometries acting effectively on M. The* underlying space *of the orbifold is M/Γ. A* bad *Riemannian orbifold is a Riemannian orbifold which does not arise as a global quotient.*

To each point $x \in U_i$ in an orbifold O is associated a group $\Gamma_x^{(i)}$, well–defined up to isomorphism within a local coordinate system: Let $U_i = \tilde{U}_i/\Gamma$ be a local coordinate system. Let $\tilde{x}, \tilde{y}$ be two points which project to x. Let $\Gamma_{\tilde{x}}^{(i)}$ be the isotropy group of $\tilde{x}$. Then if $\gamma \in \Gamma$ is the isometry such that $\gamma\tilde{x} = \tilde{y}$, it is not hard to see that the isotropy group of $\tilde{y}$ must be $\gamma\Gamma_{\tilde{x}}^{(i)}\gamma^{-1}$. Hence, the two isotropy groups are conjugate. Thus, up to isomorphism they can be regarded as the same group. We will denote this group by $\Gamma_x^{(i)}$. It can be shown (see [B] or [S2]) that $\Gamma_x^{(i)}$ up to isomorphism, is also independent of coordinate system U_i. We will denote this group unambiguously by Γ_x. Let O be a Riemannian orbifold. Let $p \in U_i \subset O$, where $U_i \overset{\text{isom}}{\cong} \tilde{U}_i/\Gamma_i$ is an open neighborhood of p. Choose $\tilde{p} \in \tilde{U}_i$ so that it projects to p. Denote the isotropy group of $\tilde{p}$ by Γ_p. Since Γ_i is finite, it is easy to see that there exists a neighborhood $U_p \subset U_i$ and corresponding $\tilde{U}_p \subset \tilde{U}_i$ such that $U_p \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$. The neighborhood U_p will be called a *fundamental neighborhood of* p. The open set $\tilde{U}_p$ will be called a *fundamental chart.*

Definition 8. *The* singular set Σ_O *of an orbifold* O *consists of those points* $x \in O$ *whose isotropy subgroup* Γ_x *is non–trivial. We say that* O *is a manifold when* $\Sigma_O = \emptyset$. *We may also, by abuse of definition, call points in the local covering* $\tilde{U}_i$ *with non–trivial isotropy, singular points also. This should cause no confusion since* $x \in O$ *is singular if and only if a corresponding point* $\tilde{x} \in \tilde{U}_i$ *is singular.*

To distinguish certain subsets of the singular set, we make the following definitions.

Definition 9. *Let* U *be a Riemannian manifold, and let* G *be a a finite group of isometries acting on* U. *Let* $H \subset G$ *be a subgroup of* G. *The subset*

$$U_H = \{x \in U \mid \Gamma_x = H\}$$

is called the stratum of U associated with H. *A* stratification *of* U *is the partitioning of* U *into strata corresponding to every subgroup of* G. *Note that under these hypotheses, any such stratification is the union of a finite number of strata.*

It can be shown that any stratum U_H associated to a subgroup $H \subset G$ is a totally geodesic submanifold of U. See [B].

Remark 10. If we define the subset $U'_H = \{x \in U \mid H \subset \Gamma_x\} \subset U$ then $U_H \subset U'_H$ and U'_H is a closed totally geodesic submanifold of U. See [Ko]. Thus, although $\overline{U_H} \subset U'_H$, $\overline{U_H} \neq U'_H$ in general as the following example shows.

Example 11. Let $U = \mathbb{R}^2$, and let $G = \langle \alpha, \beta \rangle \subset \mathbb{O}(2)$ be the group of isometries generated by α, β, where α is rotation about the origin through an angle of $\pi/2$ and β is reflection across the x–axis. If $H = \langle \alpha \rangle$, then $U_H = \emptyset$ and $U'_H = \{(0,0)\}$.

Remark 12. The proof that U'_H is a closed totally geodesic submanifold can be used to show that given any isometry g of a Riemannian manifold U, the fixed point set of g is a closed totally geodesic submanifold of U. See [Ko]. Since Riemannian orbifolds are locally (open) Riemannian manifolds modulo finite group actions, it follows that the singular set, locally, is the image of the union of a finite number of closed submanifolds of U. Since any submanifold of U has empty interior in U, we can conclude that in the case of Riemannian orbifolds, the singular set is closed and has empty interior.

In order to do Riemannian geometry on orbifolds we need to know how to measure the lengths of curves. To do this, we lift curves locally, so that we may compute their lengths locally in fundamental neighborhoods. Finally, we add up these local lengths to get the total length of the curve. The problem of course, is that locally these lifts are *not* unique. It will turn out, however, that the length of a curve is well–defined. We refer to [B] for the details. We are now in a position to give a length space structure to any Riemannian orbifold O. Given any two points x, $y \in O$ define the distance $d(x,y)$ between x and y to be

$$d(x,y) = \inf\{L(\gamma) \mid \gamma \text{ is a continuous curve joining } x \text{ to } y\}.$$

Then (O,d) becomes a length space. Furthermore if (O,d) is complete, any two points can be joined by a minimal geodesic realizing the distance $d(x,y)$. See [G]. In the case of a good Riemannian orbifold $O = (M,\Gamma)$, it follows that for x, $y \in M/\Gamma$,

$$d(x,y) = d_M\big(\mathrm{pr}^{-1}(x),\mathrm{pr}^{-1}(y)\big) \overset{\mathrm{def}}{=} \inf_{\tilde{x}\in \mathrm{pr}^{-1}(x),\, \tilde{y}\in \mathrm{pr}^{-1}(y)} d_M(\tilde{x},\tilde{y}).$$

This is because (M,d_M) is itself a length space. If M is complete, then it follows that x, y can be joined by a minimal geodesic which corresponds to the projection of the minimal geodesic realizing the distance $d_M\big(\mathrm{pr}^{-1}(x),\mathrm{pr}^{-1}(y)\big)$. It can also be shown that in this case, O is complete if and only if M is complete.

A version of Toponogov's theorem holds for Riemannian orbifolds. In [BGP] it is shown that, for instance, a locally compact length space which has Toponogov curvature $\geq k$ locally, has Toponogov curvature $\geq k$ globally. It follows that orbifolds modelled locally on Riemannian manifolds M with $K_M \geq k$ have Toponogov curvature $\geq k$.

Structure of Geodesics in Orbifolds. We now investigate the behavior of segments in orbifolds. All orbifolds in this section are assumed to be *complete* as metric spaces. The first result shows that the singular set Σ is locally convex.

Lemma 13. *Let $O = (M,\Gamma)$ be a complete, good Riemannian orbifold, and let Σ be its singular set. Given $p \in \Sigma$ there exists $\varepsilon_p > 0$ such that for all $q \in \Sigma \cap B(p,\varepsilon_p)$ any segment in O between p and q lies in Σ. Thus, Σ is locally convex.*

Proof. Note that the statement is trivial if p is an isolated point of Σ. So assume p is not isolated. Then there exists $\tilde{p} \in \mathrm{pr}^{-1}(p)$ and a neighborhood $\tilde{U}_p$ so that for sufficiently small ε_p, $B(p,\varepsilon_p) \subset U \overset{\mathrm{isom}}{\cong} \tilde{U}_p/\Gamma_p$. If necessary, choose ε_p smaller so that $2\varepsilon_p < \mathrm{inj}_{\tilde{p}}M$. Suppose to the contrary that for some $q \in B(p,\varepsilon_p) \cap \Sigma$, there exists a segment γ from p to q not entirely contained in Σ. Then there exists some point $r \in \gamma$ so that Γ_p does not fix r. By taking a small enough metric ball around q which is contained in $B(p,\varepsilon_p)$, we may assume that that $\Gamma_q \subset \Gamma_p$. Since $\#\Gamma_q > 1$, pulling γ back to M gives rise to (at least) two segments from $\tilde{p}$ to $\tilde{q}$ in M which is absurd since $\tilde{q} \in B(\tilde{p},\varepsilon_p)$ and $2\varepsilon_p < \mathrm{inj}_{\tilde{p}}M$. This completes the proof. $\square$

The next lemma assures that a segment in an orbifold minimizes distance between any two of its points.

Lemma 14. *Let $\gamma : [0,1] \to O$ be a segment and let $\tilde{\gamma} \subset M$ be a lift of γ such that $L(\tilde{\gamma}) = L(\gamma)$. Then $d_M\big(\tilde{\gamma}(t_1), \tilde{\gamma}(t_2)\big) = d_O\big(\gamma(t_1), \gamma(t_2)\big)$ for all t_1, $t_2 \in [0,1]$.*

Proof. Suppose not. Let $\gamma(t_1) = r$, $\gamma(t_2) = s$, $\tilde{\gamma}(t_1) = \tilde{r}$, $\tilde{\gamma}(t_2) = \tilde{s}$. Then

$$d_O(r,s) = d_M\big(\mathrm{pr}^{-1}(r), \mathrm{pr}^{-1}(s)\big) \neq d_M(\tilde{r}, \tilde{s})$$

by hypothesis. So, suppose that $d_M\big(\mathrm{pr}^{-1}(r), \mathrm{pr}^{-1}(s)\big)$ is realized by $\tilde{r}'$, $\tilde{s}'$ (where $\tilde{r} = \tilde{r}'$ or $\tilde{s} = \tilde{s}'$ is possible, but not both). Let g be an isometry taking $\tilde{r}'$ to $\tilde{r}$, and let h be an isometry taking $\tilde{s}$ to $g\tilde{s}'$. Consider the new path $\tilde{\gamma}' = (\tilde{p}, \tilde{r}, g\tilde{s}', h\tilde{q})$. Then $\tilde{\gamma}'$ is a shorter path than γ which projects down to a path from p to q in O, a contradiction since $d_M(\tilde{p}, \tilde{q}) = d_O(p,q)$. This completes the proof. □

The last result of this section shows that in some sense the set Σ forms a geometric barrier to length minimization.

Proposition 15. *Suppose $\gamma : [0,1] \to O$ is a segment. Let $\gamma(0) = p, \gamma(1) = q$. Then either*

(i) $\gamma \subset \Sigma$ *or*
(ii) $\gamma \cap \Sigma \subset \{p\} \cup \{q\}$

In particular, if $\gamma \not\subset \Sigma$, then $\gamma \cap \Sigma = \emptyset$, $\{p\}$, $\{q\}$, or *$\{p\} \cup \{q\}$.*

Proof. Suppose $\gamma \not\subset \Sigma$ and that $p \notin \Sigma$. Then let $r \in \gamma \cap \Sigma$, $r = \gamma(t_0)$, $t_0 \neq 0$ be the first time γ intersects Σ. Note that such a first time exists, since Σ is closed and $p \notin \Sigma$. If $t_0 = 1$, then $\gamma \cap \Sigma = \{q\}$, which is fine. So assume $t_0 \neq 1$. Now pull γ back to M and observe that there exists an isometry $g \in \Gamma_r$ which must move $\tilde{p}$. But, then we can construct a branching geodesic as follows: Note that the curve $-\tilde{\gamma} = (\tilde{q}, \tilde{r}, \tilde{p})$ has the same length as $-\tilde{\gamma}' = (\tilde{q}, \tilde{r}, g\tilde{p})$. Since γ is a segment we have

$$L(-\tilde{\gamma}') = L(-\tilde{\gamma}) = d(\tilde{p}, \tilde{q}) = d(\Gamma\tilde{p}, \Gamma\tilde{q}).$$

We therefore can conclude that $-\tilde{\gamma}'$ realizes the distance between $g\tilde{p}$ and $\tilde{q}$, and thus it is a geodesic. But this situation gives rise to a branching geodesic which is impossible in a Riemannian manifold. Finally, if $p \in \Sigma$ and $\gamma \not\subset \Sigma$ and γ does not immediately leave Σ, then, by local convexity of Σ (Lemma 13), there exists $\varepsilon > 0$ and $\delta > 0$ so that $\gamma(\varepsilon) \subset \Sigma$ for $0 \leq t \leq \varepsilon < 1$, and $\gamma(\varepsilon + \delta_0) \not\subset \Sigma$ for $0 < \delta_0 < \delta$. Then we have a curve that lies in Σ, then tries to leave momentarily. This is identical to the situation above. Thus $\gamma \subset \Sigma$ unless no such ε exists. In other words, γ immediately leaves Σ, and we conclude that γ can only intersect Σ again (possibly) at its endpoint q. The proof is now complete. □

Remark 16. Since all of the arguments of this section only used the local structure of orbifolds, all of these results hold for general orbifolds.

The significance of the last theorem is apparent. It says that a segment cannot pass through the singular set unless it starts and/or ends there. A trivial consequence of this is that the complement of Σ in O is convex in the sense that all points in $O - \Sigma$ can be joined by some segment. Thus, Σ cannot disconnect O. In particular, it follows that a Riemannian orbifold is geodesically complete if and only if it is a Riemannian manifold. Hence a Riemannian orbifold is an almost Riemannian space (see [P]) if and only if it is a Riemannian manifold.

Volume Comparison for Orbifolds. Before we define the concept of volume for a Riemannian orbifold, we need to recall the following definitions:

Definition 17. *Let X be a metric space. The σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and will be denoted by $\mathcal{B}_X$. Given a measure μ on $\mathcal{B}_X$, there is a unique measure $\bar{\mu}$ which is complete and extends μ. $\bar{\mu}$ is defined on the new σ-algebra*

$$\overline{\mathcal{B}_X} = \mathcal{B}_X \cup \{F \mid F \subset A,\ A \in \mathcal{B}_X \text{ and } \mu(A) = 0\}$$

and $\bar{\mu}(F) \overset{\text{def}}{=} 0$.

We have previously remarked that the singular set is covered locally by the union of a finite number of totally geodesic submanifolds. This union thus has measure 0 relative to the canonical Riemannian measure in each $\tilde{U}_p$. Since the natural projection to the orbifold is distance decreasing, it is natural to require that any measure constructed on the orbifold assign the singular set measure 0. Of course, we also want the orbifold measure to be compatible with the local Riemannian measures that come from the covering. This is addressed in the next lemma.

Lemma 18. *For any Riemannian orbifold O with singular set Σ, there exists a complete canonical measure $\bar{\mu}$ on $\overline{\mathcal{B}_{O-\Sigma}}$, given by a unique volume form on $O - \Sigma$. Furthermore, $\bar{\mu}$ can be extended to a complete measure $\bar{\nu}$ on $\overline{\mathcal{B}_O}$. Explicitly,*

$$\bar{\nu}(A) = \bar{\mu}(A - \Sigma) = \int_{A-\Sigma} d\mathrm{Vol}$$

for any $A \in \overline{\mathcal{B}_O}$. Here, $d\mathrm{Vol}$ is to be interpreted as $d\bar{\mu}$. In particular, $\bar{\nu}(F) = 0$ for any $F \subset \Sigma$.

Proof. Let $p \in O$, and let $U_p \cong \tilde{U}_p/\Gamma_p$ be a fundamental neighborhood of p. Let pr : $\tilde{U}_p \to U_p$ be the natural projection. Let $\tilde{\Sigma}_p = \mathrm{pr}^{-1}(\Sigma \cap U_p)$. Then on $\tilde{U}_p - \tilde{\Sigma}$, Γ_p acts properly discontinuously without fixed points. Since the action is by isometries, the canonical Riemannian volume form $\tilde{\Omega}$ on $\tilde{U}_p$ is invariant under the action of Γ_p. Hence it follows that there exists a unique volume form Ω on $U_p - \Sigma$ such that $\mathrm{pr}^*\Omega = \tilde{\Omega}$. See [BG, Lemma 5.3.9]. Since $O - \Sigma$ is connected we conclude that the volume form Ω is unique. Completing the resulting measure gives rise to a complete measure $\bar{\mu}$ on $\overline{\mathcal{B}_{O-\Sigma}}$ which is to be extended to a complete measure $\bar{\nu}$ on $\overline{\mathcal{B}_O}$. The extension is given by the formula

$$\bar{\nu}(A) = \bar{\mu}(A - \Sigma) = \int_{A-\Sigma} d\mathrm{Vol}$$

for $A \in \overline{\mathcal{B}_O}$. Then $\bar{\nu}$ is indeed complete. Note that this definition is compatible with the canonical measure in each $\tilde{U}_p$. For, $\tilde{\Sigma}_p \in \mathcal{B}_{\tilde{U}_p}$ and has measure 0 in $\tilde{U}_p$ since $\tilde{\Sigma}_p$ is the finite union of closed totally geodesic submanifolds of $\tilde{U}_p$. Next since pr is distance decreasing it must follow that $\mathrm{pr}(\tilde{\Sigma}_p) = \Sigma \cap U_p \in \mathcal{B}_O$ and has measure 0 in O. This completes the proof. □

The geodesic structure theorem of the previous section says that once a geodesic hits the singular set it must stop. Thus, in some sense the domain of the "exponential" map for an orbifold is *smaller* than its counterpart in the local Riemannian covering. Combining this with the fact that the natural projection is distance decreasing gives us, at least intuitively, reason to believe that volume cannot be concentrated *behind* singular points. It is this reasoning that enables us to now extend the Bishop relative volume comparison theorem to orbifolds, but first we need a notion of Ricci curvature.

Definition 19. *A Riemannian orbifold is said to have* $\mathrm{Ric}_O \geq (n-1)k$ *if every point is locally covered by a Riemannian manifold with Ricci curvature* $\geq (n-1)k$.

Proposition 20. *Let O be a complete Riemannian orbifold with singular set Σ. Suppose* $\mathrm{Ric}_O \geq (n-1)k$. *Then the function*

$$r \mapsto \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}_k\, B(\bar{p},r)}$$

is non-increasing. $\mathrm{Vol}_k\, B(\bar{p},r)$ *denotes the volume of the metric r-ball in S_k^n, the n-dimensional simply connected space form of constant curvature k. Furthermore, the limit as $r \to 0$ is $1/\#\Gamma_p$, where Γ_p is the isotropy subgroup at p.*

Proof. Note that $O - \Sigma$ is a (non–complete) Riemannian manifold. Fix $p \in O$. Let $\varepsilon_i \to 0$ be a sequence of real numbers, and $\{p_i\}$, a sequence of points in O such that $d(p,p_i) < \varepsilon_i$. Then clearly,

$$\lim_{i\to\infty} d_H\big(B(p_i,r),B(p,r)\big) = 0$$

where d_H denotes the usual Hausdorff distance between sets in the metric space O. It follows that

$$\text{Vol } B(p_i,r) \longrightarrow \text{Vol } B(p,r).$$

To see this, define the characteristic function $\chi_A : O \to \mathbb{R}$ for a subset $A \subset O$ to be

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

Then we have that

$$\chi_{B(p_i,r)} \to \chi_{B(p,r)}$$

pointwise almost everywhere. For, if $x \in B(p,r)$, then $d(p,x) = r - \delta$, $\delta > 0$, thus by the triangle inequality,

$$d(p_i,x) \leq d(p,p_i) + d(p,x) \leq d(p,p_i) + r - \delta.$$

Hence, if i is chosen so that $d(p,p_i) < \frac{1}{2}\delta$, then $x \in B(p_i,r)$. On the other hand, if $x \notin \overline{B(p,r)}$, then a similar argument shows that $x \notin B(p_i,r)$ for sufficiently large i. Thus, by Lebesgue dominated convergence

$$\text{Vol } B(p_i,r) = \int_{O-\Sigma} \chi_{B(p_i,r)}\, d\text{Vol} \longrightarrow \int_{O-\Sigma} \chi_{B(p,r)}\, d\text{Vol} = \text{Vol } B(p,r)$$

where $d\text{Vol}$ is the Riemannian measure on $O - \Sigma$. Since $O - \Sigma$ is convex, and we have a well–defined exponential map $\exp_{p_i}$ that is defined on the interior of $\text{Cut}(p_i) - \Sigma \subset \text{Cut}(p_i)$, where $\text{Cut}(p_i)$ denotes the cut locus at p_i, we can apply the standard volume comparison theorem to conclude that

$$\frac{\text{Vol } B(p_i,r)}{\text{Vol } B(p_i,R)} \geq \frac{\text{Vol}_k\ B(\bar{p},r)}{\text{Vol}_k\ B(\bar{p},R)}.$$

Letting $i \to \infty$ gives

$$\frac{\text{Vol } B(p,r)}{\text{Vol } B(p,R)} \geq \frac{\text{Vol}_k\ B(\bar{p},r)}{\text{Vol}_k\ B(\bar{p},R)}.$$

To prove the last statement of the theorem, consider a fundamental neighborhood $U_p \overset{\text{isom}}{=} \tilde{U}_p/\Gamma_p$. Let $r > 0$ be such that $B(\tilde{p},r) \subset \tilde{U}_p$. Choose a point $\tilde{q}$ not in the fixed point set of Γ_p and choose a Dirichlet domain $\mathcal{D}_r \subset B(\tilde{p},r)$ centered at $\tilde{q}$. Then the translates of $\mathcal{D}_r$ cover $B(\tilde{p},r)$ and have volume equal to $1/\#\Gamma_p \cdot \text{Vol } B(\tilde{p},r)$. Since from standard volume comparison we have

$$\lim_{r\to 0^+} \frac{\text{Vol } B(\tilde{p},r)}{\text{Vol}_k\ B(\bar{p},r)} = 1$$

we conclude

$$\lim_{r\to 0^+} \frac{\text{Vol}\, B(p,r)}{\text{Vol}_k\; B(\bar{p},r)} = \frac{1}{\#\Gamma_p}.$$

This completes the proof. □

Corollary 21. *Let O be a complete Riemannian orbifold with* $\text{Ric}_O \geq (n-1)$. *Then* $\text{diam}(O) \leq \pi$.

The Sphere Theorems. In order to prove an orbifold version of Cheng's theorem we will need to recall the following definitions and results.

Definition 22. *A bounded metric space (X,d) is said to have excess $\leq \varepsilon$ provided that there are points p, $q \in X$ such that $d(p,x)+d(x,q) \leq d(p,q)+\varepsilon$ for all $x \in X$. The excess, denoted $e(X)$, is the infimum over all $\varepsilon \geq 0$ such that X has excess $\leq \varepsilon$.*

Remark 23. If X is compact then there exists p, $q \in X$ such that $d(p,x) + d(q,x) \leq d(p,q) + e(X)$ for all $x \in X$.

The next lemma is a simple generalization to orbifolds of a result in [GP1]. We use the notation there: $B(p,r)$ will denote the closed metric r–ball in O centered at p, and $V(n,r)$ the volume of an r–ball in S^n of constant curvature 1.

Lemma 24. *Let O be a complete Riemannian orbifold with* $\text{Ric}_O \geq (n-1)$ *and* $\text{diam}(O) = D$. *If p, $q \in O$ with $d(p,q) = D$ and $\alpha+\beta = D$ then $O = B(p,\alpha+\varepsilon) \cup B(q,\beta+\varepsilon)$ whenever $V(n,\varepsilon) \geq V(n,D) - 2V(n,\frac{1}{2}D)$. In particular, $e(O) \leq 2\varepsilon$.*

Proof. Given the relative volume comparison theorem for orbifolds, the proof is the same as in [GP1]. □

Remark 25. It follows that Riemannian orbifolds with $\text{Ric}_O \geq (n-1)$ and diameters close to π have small excess. In particular if $\text{diam}(O) = \pi$, then $e(O) = 0$.

We will also need the following result of Grove–Petersen [GP2].

Lemma 26. *Let X be a compact length space with $e(X) = 0$ and curvature bounded from below in the sense of Toponogov, then X is a topological suspension.*

Definition 27. *Let X be a length space with Toponogov curvature ≥ 1. Then the* sin–suspension, *$\Sigma_{\sin} X$ of X is the topological suspension,*

$$\Sigma X = X \times [0,\pi] \Big/ X \times \{0,\pi\}$$

equipped with the following metric. Let (x,t), (y,s) be two points of ΣX, then

$$d\big((x,t),(y,s)\big) \stackrel{\text{def}}{=} d_{S^2}\big(\gamma_1(t), \gamma_2(s)\big)$$

where γ_i are great circle arcs parametrized by arclength, with $\gamma_1(0) = \gamma_2(0)$ and $\angle\big(\dot\gamma_1(0), \dot\gamma_2(0)\big) = d_X(x,y)$. $\Sigma^m_{\sin} X$ will denote the m–fold sin*–suspension*

$$\Sigma^m_{\sin} X = \underbrace{\Sigma_{\sin} \dots \Sigma_{\sin}}_{m} X.$$

Remark 28. If X is a complete Riemannian manifold with $\mathrm{Ric}_X \geq (n-1)$ then it follows from general formulas for a Riemannian warped products that the radial curvatures of $\Sigma_{\sin} X$ are $\equiv 1$. See [BO] and [GP2]. Also there is a notion of sin–suspensions over general length spaces, but even if the length space has Toponogov curvature $\geq k$, $k < 0$, the resulting suspension will not have Toponogov curvature $\geq k$ for any $k \in \mathbb{R}$. For example, let $T^2 = S^1 \times S^1$ be the flat torus. Then $\Sigma_{\sin} T^2$ does not have Toponogov curvature $\geq k$ for any $k \in \mathbb{R}$. See [BGP].

Lemma 29 (Grove–Petersen)**.** *Let X be a complete length space with Toponogov curvature ≥ 1 and diameter $= \pi$. Then $e(X) = 0$ and is realized by two points p, q with $d(p,q) = \pi$. Moreover, if $E = \{x \in X \mid d(p,x) = d(q,x) = \frac{1}{2}\pi\}$, then X is isometric to $\Sigma_{\sin} E$.*

Proof. See [GP2].

Definition 30. *A n–*dimensional orbifold space form of constant curvature *k is a good orbifold (M,Γ), where $M \stackrel{\text{isom}}{=} S^n_k$, the n–dimensional simply connected Riemannian space form of constant curvature k. If $n = 0$, there are exactly two such orbifold space forms, namely, the metric space consisting of exactly two points $\{x,y\}$ with $d(x,y) = \pi/\sqrt{k}$ and the metric space consisting of a single point. Note that technically these two metric spaces can be regarded as 0–dimensional Riemannian space forms.*

The next proposition is a kind of analogue of the Grove–Shiohama [GS] sphere theorem.

Lemma 31. *Let O be an n–dimensional space form of constant curvature 1. If* $\mathrm{diam}(O) < \pi$*, then, in fact,* $\mathrm{diam}(O) \leq \frac{1}{2}\pi$*.*

Proof. Assume $\frac{1}{2}\pi < \operatorname{diam}(O) < \pi$. Let p, q be such that $d(p,q) = \operatorname{diam}(O)$. Then $d(\mathrm{pr}^{-1}(p), \mathrm{pr}^{-1}(q)) > \frac{1}{2}\pi$. In particular, the finite set $\mathrm{pr}^{-1}(p) = \{\tilde{p}_1, \dots, \tilde{p}_m\}$ lies entirely in an open hemisphere H. We can construct the center of mass $\tilde{p}_c' \in \mathbb{R}^{n+1}$ of the set $\mathrm{pr}^{-1}(p)$, namely,

$$\tilde{p}_c' = \frac{\tilde{p}_1 + \ldots + \tilde{p}_m}{m}.$$

Then $\tilde{p}_c'$ lies in the convex hull of $\mathrm{pr}^{-1}(p)$ and hence lies in the open half-space containing H. Thus, $\tilde{p}_c'$ projects to a unique point $\tilde{p}_c \in H \subset S^n$. Since the center of mass $\tilde{p}_c'$ is fixed by Γ, $\tilde{p}_c$ is fixed. Its antipode $-\tilde{p}_c$, must also be fixed. Thus $\operatorname{diam}(O) = \pi$, which is a contradiction. This completes the proof. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Myers' theorem, $\operatorname{diam}(M) \leq \pi$. Since $\operatorname{diam}(O) = \pi$, there exists a segment in M of length π, so $\operatorname{diam}(M) = \pi$. By Cheng's maximal diameter theorem, it follows that $M \overset{\text{isom}}{=} S^n$, the sphere of constant curvature 1. Choose p, $q \in O$ with $d(p,q) = \pi$, and let γ be a segment joining them. This segment then lifts to a great circle arc on $M = S^n$. Denote the preimages of p, q by $\tilde{p}$, $\tilde{q}$ respectively. Observe that each element of Γ must fix both $\tilde{p}$, $\tilde{q}$. To see this, suppose that $\tilde{p}$ is not fixed by some element $g \in \Gamma$. Let $g\tilde{p} = \tilde{p}'$. Note $\tilde{p}' \neq \tilde{q}$. Thus, the piece of great circle arc joining $\tilde{p}'$ to $\tilde{q}$ which has length $< \pi$, pushes down to a curve in O of length $< \pi$ connecting p to q, which is a contradiction. Thus, every element of Γ must fix $\tilde{p}$ and $\tilde{q}$. Let $N = \{\tilde{x} \in M \mid g\tilde{x} = \tilde{x} \ \ \forall g \in \Gamma\}$. Then $N \subset M$ is a closed totally geodesic submanifold containing $\tilde{p}$ and $\tilde{q}$. Hence $\operatorname{diam}(N) = \pi$ and N satisfies the curvature hypothesis of Cheng's theorem since it is totally geodesic. Thus, $N \overset{\text{isom}}{=} S^k$ for some $0 \leq k < n$. Here we define S^0 of constant curvature 1 to be the two element metric space $\{x,y\}$ with $d(x,y) = \pi$, and S^1 of constant curvature 1 to be the circle of radius 1 contained in $\mathbb{R}^2$. Now, O satisfies the hypothesis of Lemma 29 by applying the Toponogov theorem for orbifolds. Hence, $O = \Sigma_{\sin} E$, where $E = \{x \in O \mid d(p,x) = d(q,x) = \frac{1}{2}\pi\}$. Note that $\mathrm{pr}^{-1}(E) = S^{n-1} \subset S^n$, the equator relative to $\tilde{p}$ and $\tilde{q}$. To see this, suppose $x \in E$. Choose $\tilde{x} \in \mathrm{pr}^{-1}(x)$ so that $d(\tilde{p}, \tilde{x}) = \frac{1}{2}\pi$. But then, since Γ fixes $\tilde{p}$, $d(\tilde{p}, \mathrm{pr}^{-1}(x)) = \frac{1}{2}\pi$, which implies that $\mathrm{pr}^{-1}(x) \subset S^{n-1}$. Now suppose $\tilde{x} \in S^{n-1}$. Then $\frac{1}{2}\pi = d(\tilde{p}, \tilde{x}) = d(g\tilde{p}, g\tilde{x}) = d(\tilde{p}, g\tilde{x})$ for all $g \in \Gamma$. Thus, $\mathrm{pr}(\tilde{x}) \in E$ and hence $\mathrm{pr}^{-1}(E) = S^{n-1}$. Observe that S^{n-1} is invariant under Γ. The problem now reduces to two cases: (1) $N = S^0$, and (2) $N = S^k$, $0 < k < n$. In case (1), just observe that by definition of N no point of S^{n-1} is fixed by every element of Γ. Hence, $E \overset{\text{isom}}{=} S^{n-1}/\Gamma$ is a $(n-1)$-dimensional orbifold space form of constant curvature 1, and $\operatorname{diam}(E) < \pi$. The argument that the diameter must be less than π is the same as in the beginning of this proof. By Lemma 31, $\operatorname{diam}(E) \leq \frac{1}{2}\pi$. For case (2), Take $S = S^1 \subset N = S^k$ to be any great circle $\subset N$ containing $\tilde{p}$ and $\tilde{q}$. Then $\{\tilde{x}, \tilde{y}\} = S \cap S^{n-1}$ are fixed by Γ, and hence $E = \mathrm{pr}(S^{n-1})$ has $\mathrm{diam}(E) = \pi$. Finally, since S^{n-1} is invariant under Γ, we can proceed by induction to get the conclusion of the theorem. This completes the proof. □

Remark 32. Note the natural inclusion of $\mathbb{O}(n) \subset \mathbb{O}(n+1)$ naturally extends any isometric group action on S^{n-1} to an isometric action on S^n, in which the original action is now an action on an equator of S^n. This induced group action fixes the two antipodal points of S^n which lie on the line in $\mathbb{R}^{n+1}$ perpendicular to this equator. The resulting n–dimensional orbifold space form must be a sin–suspension over E, the equatorial quotient, by Lemma 29. Hence, we can conclude that the sin–suspension of an orbifold space form is again an orbifold space form.

We now prove Theorem 1.

Proof of Theorem 1. The first step in the proof is to show that in some sense O has maximal volume growth. Choose $p, q \in O$ with $d(p,q) = \pi$, and two antipodal points $\bar{p}, \bar{q} \in S^n$. Then

$$B\left(p, \frac{1}{2}\pi\right) \cap B\left(q, \frac{1}{2}\pi\right) = \emptyset$$

and thus,

$$\text{(1)} \qquad \mathrm{Vol}\, O \geq \mathrm{Vol}\, B\left(p, \frac{1}{2}\pi\right) + \mathrm{Vol}\, B\left(q, \frac{1}{2}\pi\right).$$

By volume comparison we have

$$\frac{\mathrm{Vol}\, O}{\mathrm{Vol}\, S^n} = \frac{\mathrm{Vol}\, B(p,\pi)}{\mathrm{Vol}\, B(\bar{p},\pi)} \leq \frac{\mathrm{Vol}\, B(p,\frac{1}{2}\pi)}{\mathrm{Vol}\, B(\bar{p},\frac{1}{2}\pi)}.$$

Therefore, $\frac{1}{2}\mathrm{Vol}\, O \leq \mathrm{Vol}\, B(p, \frac{1}{2}\pi)$. A similar inequality holds for q. This means we have equality in (1) and that $\mathrm{Vol}\, B(p, \frac{1}{2}\pi) = \mathrm{Vol}\, B(q, \frac{1}{2}\pi) = \frac{1}{2}\mathrm{Vol}\, O$. Hence

$$\frac{\mathrm{Vol}\, O}{\mathrm{Vol}\, S^n} = \frac{\mathrm{Vol}\, B(p,\frac{1}{2}\pi)}{\mathrm{Vol}\, B(\bar{p},\frac{1}{2}\pi)} = \frac{\mathrm{Vol}\, B(q,\frac{1}{2}\pi)}{\mathrm{Vol}\, B(\bar{q},\frac{1}{2}\pi)}.$$

Then for $r \in [\frac{1}{2}\pi, \pi)$, we have

$$\begin{aligned}
\text{(2)} \qquad & \mathrm{Vol}\, O \geq \mathrm{Vol}\, B(p,r) + \mathrm{Vol}\, B(q,\pi - r) \\
= \quad & \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}\, B(\bar{p},r)}\mathrm{Vol}\, B(\bar{p},r) + \frac{\mathrm{Vol}\, B(q,\pi - r)}{\mathrm{Vol}\, B(\bar{q},\pi - r)}\mathrm{Vol}\, B(\bar{q},\pi - r) \\
\overset{\substack{\text{volume} \\ \text{comparison}}}{\geq} \quad & \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}\, B(\bar{p},r)}\mathrm{Vol}\, B(\bar{p},r) + \frac{\mathrm{Vol}\, O}{\mathrm{Vol}\, S^n}\mathrm{Vol}\, B(\bar{q},\pi - r) \quad \geq
\end{aligned}$$

$$\stackrel{\text{vol. comp}}{\geq} \frac{\text{Vol } B(p,\pi)}{\text{Vol } B(\bar{p},\pi)}\text{Vol } B(\bar{p},r) + \frac{\text{Vol } O}{\text{Vol } S^n}\text{Vol } B(\bar{q},\pi - r)$$

$$= \frac{\text{Vol } O}{\text{Vol } S^n}\text{Vol } S^n = \text{Vol } O.$$

Thus,

$$\text{Vol } O = \text{Vol } B(p,r) + \text{Vol } B(q,\pi - r). \tag{3}$$

Hence we have equality in (2). So by (3), we have

$$\text{Vol } O = \underbrace{\left(\text{Vol} O - \text{Vol} B(q,\pi - r)\right)}_{=\text{Vol } B(p,r)} + \frac{\text{Vol } O}{\text{Vol } S^n}\text{Vol } B(\bar{q},\pi - r).$$

This implies that

$$\frac{\text{Vol } O}{\text{Vol } S^n} = \frac{\text{Vol } B(q,\pi - r)}{\text{Vol } B(\bar{q},\pi - r)}.$$

Now let $r \to \pi$, then we see that

$$\frac{1}{\#\Gamma_q} = \frac{\text{Vol } O}{\text{Vol } S^n}.$$

By interchanging the role of p and q we conclude that

$$\text{Vol } B(p,r) = \frac{1}{\#\Gamma_p}\text{Vol } B(\bar{p},r) \quad \text{for } r \in [0,\pi].$$

The next step is to show that O is isometric to a quotient of (S^n,can). To do this, let $U_p \cong \tilde{U}_p/\Gamma_p$ be a fundamental neighborhood of p. Let $\tilde{p} = \text{pr}^{-1}(p)$ and choose $r > 0$ so that $B(\tilde{p},r) \subset \tilde{U}_p$. Let $T_{\tilde{p}}\tilde{U}_p$ be the tangent space at $\tilde{p}$. Consider the following diagram:

$$\begin{array}{ccc} T_{\tilde{p}}\tilde{U}_p \supset B(0,r) & \xrightarrow{\exp_{\tilde{p}}} & \tilde{U}_p \\ \text{pr}\big\downarrow & & \big\downarrow\text{pr} \\ T_{\tilde{p}}\tilde{U}_p/d\Gamma_p \supset B(0,r)/d\Gamma_p & \xrightarrow{\overline{\exp_{\tilde{p}}}} & U_p = \tilde{U}_p/\Gamma_p \end{array}$$

where $\overline{\exp_{\tilde{p}}}$ is defined as follows: Let $v \in T_{\tilde{p}}\tilde{U}_p$, with $\|v\| = 1$. Let $\tilde{\gamma}_v(t)$ be the unique geodesic in $\tilde{U}_p$ with $\tilde{\gamma}_v(0) = \tilde{p}$ and $\dot{\tilde{\gamma}}_v(0) = v$. Then define

$$\overline{\exp_{\tilde{p}}}([tv]) = \text{pr} \circ \exp_{\tilde{p}}(tv) = \text{pr} \circ \tilde{\gamma}_v(t) \stackrel{\text{def}}{=} \gamma_v(t) \quad \text{for } t \in [0,r).$$

The map is well–defined. For if t, $s \in [0,r)$ and v, $w \in T_{\tilde{p}}\tilde{U}_p$ with $\|v\| = \|w\| = 1$ are such that $\mathrm{pr}(tv) = \mathrm{pr}(sw)$, then there exists an isometry dg, $g \in \Gamma_p$ so that $dg(tv) = sw$. But this implies that $t = s$ and that $g(\tilde{\gamma}_v(t)) = \tilde{\gamma}_w(t)$. Hence $\gamma_v(t) = \gamma_w(t)$ for $t \in [0,r)$. Thus, the diagram commutes. Since O is compact, it has curvature bounded below in the sense of Toponogov. Thus, by Lemma 24 and Lemma 26, O is a suspension. Hence, each $\gamma_v(t)$ has a unique geodesic extension for $t \in [0,\pi]$. This gives rise to a homeomorphism

$$\overline{\exp_{\tilde{p}}} : B(0,\pi)/d\Gamma_p \longrightarrow O - \{q\}.$$

Now let Σ_p be the singular set of $B(0,\pi)/d\Gamma_p$ and let Σ_O be the singular set of O. We claim that

$$\overline{\exp_{\tilde{p}}}(\Sigma_p - \{0\}) = \Sigma_O - \{p,q\}.$$

To see this, suppose that for some $g \in \Gamma_p$, $dg(v) = v$ for some v with $0 < \|v\| < \pi$. Let $R = \frac{1}{2}r$. Then

$$dg\left(R\frac{v}{\|v\|}\right) = R\frac{v}{\|v\|}$$

which implies that

$$g\tilde{\gamma}_{v/\|v\|}(R) = \tilde{\gamma}_{v/\|v\|}(R).$$

This says that $\gamma_{v/\|v\|}(R) \in \Sigma_O$. But by the structure theorem for geodesics this implies that $\overline{\exp_{\tilde{p}}}(v) = \gamma_{v/\|v\|}(\|v\|) \in \Sigma_O$. Similar reasoning shows that $\overline{\exp_{\tilde{p}}}|_{\Sigma_p - \{0\}}$ maps onto $\Sigma_O - \{p,q\}$. From this it follows that

$$\overline{\exp_{\tilde{p}}} : B(0,\pi)/d\Gamma_p - \big(\Sigma_p \cup \{0\}\big) \longrightarrow O - \big(\Sigma_O \cup \{p,q\}\big)$$

is a diffeomorphism between smooth manifolds. Now pull back the metric on $O - \{q\}$ to $B(0,\pi)/d\Gamma_p$. If we denote by $\overline{B(0,\pi)}$ the metric closure of $B(0,\pi)$ in this new metric, then $\overline{B(0,\pi)} \overset{\text{homeo}}{\cong} S^n$ and $\overline{\exp_{\tilde{p}}}$ extends to a global isometry of $\overline{B(0,\pi)}/d\Gamma_p$ which is Riemannian on the complement of the singular set by our previous considerations. Thus, $\overline{\exp_{\tilde{p}}}$ preserves the volumes of metric balls. We can therefore conclude that

$$\mathrm{Vol}\big(B(0,s)/d\Gamma_p\big) = \frac{1}{\#\Gamma_p}\mathrm{Vol}\ B(\bar{p},s)$$

for $s \in [0,\pi]$. Now using pr^{-1} to pull back the metric on $B(0,\pi)/d\Gamma_p$ to $B(0,\pi)$ we have (since Γ_p preserves metric balls)

$$\mathrm{Vol}\ B(0,s) \geq \#\Gamma_p \mathrm{Vol}\big(B(0,s)/d\Gamma_p\big) = \mathrm{Vol}\ B(\bar{p},s).$$

Thus, Vol $B(0,s)$ is maximal. Since the complement of the singular set in $B(0,\pi)$ is star–shaped, it follows by a standard Jacobi fields comparison argument that $B(0,\pi) - \Sigma_p$ has constant curvature 1. But we know there exists a *Riemannian* metric on $B(0,\pi)$ with constant sectional curvature 1. It thus follows that $\overline{B(0,\pi)} = (S^n, \text{can})$ and hence that

$$\overline{\exp_{\tilde{p}}} : (S^n, \text{can})/d\Gamma_p \longrightarrow O$$

is an isometry. This exhibits O as a good orbifold and the proof is complete. □

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Received: March 30th, 1992.